PETER MITTELSTAEDT

THE MODAL LOGIC OF QUANTUM LOGIC

0. INTRODUCTION

Modal logic is concerned with the concepts of *necessity* and *possibility* and a certain class of object propositions. In this paper we develop the basic concepts of a modal logic which is related to propositions about quantum physical objects. Since the object logic of quantum mechanical propositions is given by the calculi of quantum logic, the structure investigated in this paper will be called the modal logic of quantum logic.

The object language and logic of quantum physical propositions is developed here within the dialogic approach to quantum logic [1]. On the basis of these object-linguistic structures we investigate the language of meta-propositions which state the material or formal truth of object-propositions. Applying again the dialogic technique to meta-propositions the important notion of a formally true meta-proposition can be defined. Using this concept it turns out that the formal logic of meta-propositions is equivalent to the corresponding structure in ordinary logic, i.e., to the effective (intuitionistic) logic.

The modalities "necessary" and "possible" are introduced here in the framework of a meta-linguistic interpretation, which considers the modalities as statements about the object-propositions under discussion [9]. On this basis it is found that meta-propositions which state the material truth of a special class of object propositions may be considered as *quantum logical modalities* (Section 2). A detailed investigation then shows that these quantum logical modalities are intimately related to the important quantum mechanical concepts of *commensurability* and *objectivity*. An illustration of the quantum logical modalities by relations between projection operators in Hilbert space concludes this part.

In the third section we introduce the concept of a *formally true modality* which leads to the modal logic of quantum logic. It is shown that this logic of quantum logical modalities contains all the quantum logical restrictions – which come from the possible incommensurability of quantum physical

Journal of Philosophical Logic 8 (1979) 479–504. 0022–3611/79/0084–0479$02.60.
Copyright © 1979 by D. Reidel Publishing Co., Dordrecht, Holland, and Boston, U.S.A.

propositions – and thus differs essentially from the known modal calculi of ordinary logic [6]. This difference is illustrated here by some examples of modalities which hold in ordinary modal logic but not in the modal logic of quantum logic. Concluding this section the modal version of the *principle of excluded middle* is investigated and found to be of special importance for the difference between classical logic and quantum logic.

The present approach to the relation between modal logic and quantum logic which is based on a proof-theoretical and metalinguistic definition of the modalities must be clearly distinguished from the results obtained by Dishkant [19], Goldblatt [20] and Dalla Chiara [21]. These authors established a modal semantic for various formal systems of object quantum logic, which is in the spirit of Kripke's semantic for intuitionistic logic [8, 22]. In contrast to the present attempt to elaborate first of all the calculus of quantum modal logic, these investigations start from the well-known standard modal logic [6] and use it for the construction of a modal semantic for quantum logic.

1. THE LANGUAGE OF QUANTUM PHYSICS

1.1. *The object language*

In order to establish the *object language* of quantum physics, we begin with a quantum physical system S (atom, nucleus, elementary particle) and consider propositions $A, B, \ldots$ about this system S, which can be proved or disproved by measuring processes. These propositions will be called *elementary propositions*. We will assume here that elementary propositions are *value-definite*, i.e., there always exists an experimental testing procedure which decides between truth and falsity of the respective proposition.

The logical connectives $A \wedge B$ (A and B), $A \vee B$ (A or B), $A \rightarrow B$ (if A then B) and $\neg A$ (not A) will be defined by means of *dialogs*. A dialog is a formalized kind of discussion between two participants, the proponent P who asserts a certain proposition and the opponent O who attempts to refute it (for details, cf. [1], Chapt. III). The concept of dialog can be defined by general frame-rules (cf. [1], p. 51), which are briefly summarized as follows:

> After assertion of the *initial argument* by P the dialog consists of a series of *arguments* which are stated in turn by

> *P* and *O* and which are either attacks on or defences of previous arguments. Attacks are rights which may be taken at any position in the dialog; defences are commitments which must be carried out in the reverse order of the corresponding attacks and at the latest when there is no longer a right for an attack. If one of the participants has no argument to continue, he loses and the other one wins the dialog.

In addition to the frame-rules the logical connectives are defined by the possibilities of attack and defence in a dialog. Together with the elementary propositions we thus obtain the following attack-and-defence scheme:

Proposition	Symbol	Possibility of attack	Possibility of defence
Elementary proposition	A	$A?$	$A!$
Conjunction	$A \wedge B$	1?, 2?	A, B
Disjunction	$A \vee B$	?	A, B
Material implication	$A \rightarrow B$	A	B
Negation	$\neg A$	A	

Here we have denoted by $A?$ the challenge to prove A and by $A!$ the successful proof of A. The logical connectives can be used in order to extend the set of elementary propositions $\mathscr{S}_e$ by incorporating arbitrary iterations of the logical connectives. The resulting set will be denoted by $\mathscr{S}$. Formally the logical connectives $\wedge, \vee, \rightarrow$ and $\neg$ are two and one-place operations respectively on the set $\mathscr{S}$.

We will frequently use a schematic representation of dialogs, the rules of which can be briefly summarized as follows (for details cf. [1], p. 52):

> A dialog is represented by means of two columns. The left column consists of all arguments of *O*, the right one of all arguments of *P*. The succession of arguments is enumerated by rows which begin at zero. The arguments of *P* and *O* in row zero are the initial argument and the empty

argument [ ] respectively. If an argument is an attack against or a defence of an assertion stated in row x we write (x) or $\langle x \rangle$ respectively on the right hand side of the argument. If a participant does not defend, the empty argument is placed in the respective row.

Examples for dialog-schemes can be found in [1] and in Section 3 of this paper.

The length of the dialogs about the connectives are so far unrestricted since attacks and defences can be repeated an arbitrary number of times. It could be limited to a finite length if – e.g., in the dialog about $A \wedge B$ – there would be any guarantee that after a successful proof of A and B the result of the A-proof is still *available* and thus a repetition of the A-proof would lead to the same result. In order to test whether the proof result of a proposition A is still available after a proof of B, we introduce availability propositions $k(A, B)$ (commensurability) and $\bar{k}(A, B)$ (incommensurability) which state whether A is available after a B-proof or not. If these availability propositions are incorporated as additional possibilities into the attack and defence scheme of the logical connectives, the dialogs about the connectives can be limited to a finite length [1, 2].

The possibilities of attacking and defending elementary propositions, logical connectives and availability propositions are summarized in the *material dialog game* D_m. This game is called "material" since the respective dialogs contain the (non-dialogical) "material" proofs of elementary and availability propositions. Hence it is obvious that the success of a dialog about A generally depends on the elementary propositions and the availability propositions contained in it, and on the individual system S and its properties. If at the system considered the proposition W has been ascertained before the dialog, we call W the "preparation" of S and write $S = S(W)$.

If a proposition A can be successfully defended in a material dialog for a system $S = S(W)$ we write $S(W) \underset{D_m}{\vdash} A$. If the success of the dialog about A does not depend on the individual system $S^i(W)$ and only on the preparation W, we write $W \underset{D_m}{\vdash} A$. In this case we say that there exists a *strategy of success for A relative to W* within the material dialog game D_m. Moreover if a compound proposition A can be defended successfully in *every* material dialog – irrespective of the preparation – we say that there

exists a *strategy of success for* A within D_m and write $\vdash_{D_m} A$. The existence of a strategy of success for A within D_m depends in general on the material propositions contained in A and the contingent relations among them.

There are, however, compound propositions C which can successfully be defended in a material dialog irrespective of the elementary propositions contained in it and irrespective of the availability propositions $k(A, B)$ and $\bar{k}(A, B)$ of arbitrary subpropositions A and B of C. These propositions C are called *formally true* propositions. The proof procedure of these propositions can be summarized in the *formal dialog game* D_f such that a proposition C can be successfully defended in a formal dialog if and only if C is formally true. In this case we write $\vdash_{D_f} C$. In particular if a material implication $A \to B$ is formally true, we also say that the *implication* $A \leqslant B \rightleftharpoons \vdash_{D_f} A \to B$ holds. Using the symbol V for the "formally true" proposition the formal truth of a proposition C can be expressed by the equivalent formulae $\vdash_{D_f} C \rightleftharpoons \vdash_{D_f} \mathrm{V} \to C \rightleftharpoons \mathrm{V} \leqslant C$. – (The symbol "$\rightleftharpoons$" is used throughout this paper for definitions.)

The formally true propositions C form the effective quantum logic and can be comprehended in the calculus Q_{eff} (of effective quantum logic). The connection between the calculus Q_{eff} and the dialog game D_f is then given by the statement that the implication $A \leqslant B$ holds if and only if the formula "$A \leqslant B$" can be derived in Q_{eff}, which will be denoted here by $\vdash_{Q_{eff}} A \leqslant B$ [2]. If the value definiteness of the elementary propositions is taken into account, the calculus Q_{eff} can be extended by incorporating the principle of excluded middle for all propositions. Thus one arrives at the calculus Q of full quantum logic [3].

1.2. *The meta-language*

Statements about the truth or the falsity of an object proposition with respect to a given dialog-game will be called *meta-propositions*. According to the dialog-games D_m and D_f respectively we have two kinds of meta-propositions

$\vdash_{D_m} A$: "A is materially true"

and

$\vdash_{D_f} A$: "A is formally true".

The falsity of a proposition A can always be expressed by the corresponding truth of the negation $\neg A$. These statements, i.e., $\vdash_{D_m} A$ and $\vdash_{D_f} A$, will be called *elementary meta-propositions*, the proof of which has to be carried out within the framework of the object-dialog-games D_m and D_f.

In order to further extend our meta-language, we introduce the *meta-dialog-game* $\bar{D}$ which is again governed by the frame-rules mentioned above but which deals with meta-propositions instead of object-propositions $A \in \mathscr{S}$. Meta-propositions will be denoted by $\bar{A}, \bar{B}, \bar{C}, \ldots$. In the next step we then define the meta-logical connectives by means of the possibilities of attack and defence in the meta-dialog-game $\bar{D}$. The dialogic definition of the meta-connectives, which will be denoted by $\barwedge$, $\barvee$, $\bar{\Rightarrow}$ and $\bar{\neg}$, is in complete analogy to the definition of the connectives in the object language, namely by the following attack-and-defence scheme, which also incorporates the elementary meta-propositions:[1]

Meta-proposition	Symbol	Attack	Defence
Elementary meta-proposition	$\bar{A}$	$\bar{A}?$	$\bar{A}!$
Meta-conjunction	$\bar{A} \barwedge \bar{B}$	1?, 2?	$\bar{A}, \bar{B}$
Meta-disjunction	$\bar{A} \barvee \bar{B}$	?	$\bar{A}, \bar{B}$
Material meta-implication	$\bar{A} \bar{\Rightarrow} \bar{B}$	$\bar{A}$	$\bar{B}$
Meta-negation	$\bar{\neg} A$	$\bar{A}$	

In a similar way as in the object language the meta-connectives $\barwedge$, $\barvee$, $\bar{\Rightarrow}$ and $\bar{\neg}$ can be used in order to extend the set $\bar{\mathscr{S}}_e$ of elementary meta-propositions by incorporating arbitrary iterations of meta-connectives. The resulting set $\bar{\mathscr{S}}$ of meta-propositions is then given by the following inductive scheme:

(I) Elementary meta-propositions $\bar{A} \in \bar{\mathscr{S}}_e$ are meta-propositions.

(II) If $\bar{A}$ and $\bar{B}$ are meta-propositions, i.e., $\bar{A} \in \bar{\mathscr{S}}, \bar{B} \in \bar{\mathscr{S}}$, then $\bar{A} \barwedge \bar{B}, \bar{A} \barvee \bar{B}, \bar{A} \bar{\Rightarrow} \bar{B}$ and $\bar{\neg} \bar{A}$ are meta-propositions.

Elementary meta-propositions $\bar{A}$ are of the form $\vdash_{D_m} A$ or $\vdash_{D_f} A$ and assert the existence of a strategy of success for proposition A in D_m or D_f,

respectively. In both cases it is evident that elementary meta-propositions are mutually commensurable, i.e., $k(\bar{A}, \bar{B})$ is valid for all meta-propositions of this kind. It is a consequence of this commensurability property that in contrast to the object-language dialogs the respective meta-dialogs for the connectives can always be restricted to a finite number of steps. Consequently all compound meta-propositions containing one connective are commensurable as well. Hence compound meta-propositions which contain more than one meta-connective can again be defined by meta-dialogs with a finite number of steps. In this way one arrives at the important result that all finitely compounded meta-propositions are mutually commensurable and thus unrestrictedly available in a meta-dialog.

Analogous to the situation in the object language, one can define formally true meta-propositions. The totality of all formally true meta-propositions then constitute the *formal meta-logic of quantum logic*. Since meta-propositions are mutually commensurable, one thus obtains the logic of unrestrictedly available propositions, i.e., the ordinary effective logic. The principle of excluded middle can generally be proved here no more than in the well known dialogic justification of ordinary logic (cf. [5], p. 66 ff.). Hence we find that the formal meta-logic of quantum logic agrees with the ordinary effective (intuitionistic) logic, which can without contradiction be extended by incorporating the principle of excluded middle to the classical logic (cf. [5], p. 67, ff.).

2. THE QUANTUM LOGICAL MODALITIES

2.1. *The meaning of modalities in quantum physics*

In the framework of the meta-language discussed in Section 1.2 we can define a special class of meta-propositions, which will be called the *quantum logical modalities*. The reason for this terminology is that these meta-propositions have a strong similarity with the *logical modalities* "necessary" and "possible", which are known from the literature [6–9]. Starting from the definition of these quantum logical modalities a calculus can be established which will be called *the modal logic* $M(Q_{\text{eff}})$ *and* $M(Q)$ *of effective and full quantum logic* respectively. These calculi are found to be quite different from the modal calculi of ordinary logic, e.g., the systems M (G. H. v. Wright) and S4 (C. I. Lewis) (cf., e.g., [6]).

A situation in which modalities generally can be meaningfully applied is given if with respect to a certain object-system S some knowledge W is available which, however, is not sufficient in order to determine all possible properties of the system. As an example we consider the following situation in classical physics: At time $t = t_0$ the state W_0 of system S is completely known, but the laws of nature Γ_i which govern the temporal development of S are only partially available. Then it is not possible to determine all measurable properties P_i of S at a time $t > t_0$.

The initial value problem which allows for a calculation of $P_i(t)$ from the initial values contained in W_0 is not completely determined. However it could happen that a special property $P_n(t)$ can be determined by means of the laws Γ_i which are available. In this case we use the terminology "$P_n(t)$ is necessary with respect to W_0". If, on the other hand, for some $P_k(t)$ it cannot be shown that $P_k(t)$ is false, $P_k(t)$ will be said to be "possible with respect to W_0".[2]

Another example for the applicability of the modalities is a quantum mechanical system S. In general the knowledge W about a system S can be expressed by a property $\mathscr{P}_W$, which corresponds to a projection operator P_W which projects onto a subspace M_W of the Hilbert space $\mathscr{H}(S)$ of the system. Even if M_W is given by a one-dimensional subspace M_ϕ of $\mathscr{H}(S)$, in which case W is a *maximal* information about S, it will not be possible to determine the values of *all* measurable properties of S from this information ϕ by means of the laws $\Gamma_i^{(q)}$ of quantum theory. This is only the case for those propositions A, the projection operator P_A of which commutes with the projection operator P_ϕ of Φ, i.e., for which the relation $[P_\phi, P_A]_- = 0$ holds. Propositions A of this kind are also called *objective* with respect to ϕ (cf. [13], p. 93 ff.). For a proposition B with $[P_\phi, P_B] \neq 0$ the measuring value (0 or 1) of the property $\mathscr{P}_B$ cannot be determined from ϕ. In quantum theory we are confronted with a situation in which all laws $\Gamma_i^{(q)}$, i.e., the relations $M_A \leqslant M_B$ between subspaces of the Hilbert space $\mathscr{H}(S)$ are known (in principle) but in which the maximal information ϕ about S is not complete. It is well known that it is impossible to enlarge a maximal information ϕ to a complete one.

It is obvious that the situation in quantum theory is particularly suited for the application of the modalities. If for a system $S(W)$ with the property $\mathscr{P}_W$ the truth of a proposition A (i.e., the eigenvalue 1 of P_A) can be determined from W with certainty, this proposition A will be called

"necessary with respect to W". If on the other hand, the falsity of a proposition A (i.e., the eigenvalue O of P_A) cannot be determined from W with certainty, this proposition A will be said to be "possible with respect to W" [14]. It is important to note that the concepts of "necessity" and "possibility" defined here are always related to the totality of contingent laws $\Gamma_i^{(q)}$ of quantum theory.

At first glance the modal interpretation of futuric propositions in classical physics and of measurable properties of quantum mechanical systems seems to be quite analogous. In the classical case we have assumed that the initial state of the system in question is completely known but the contingent physical laws which govern the temporal development of the system are only partially known. Hence the final state is not fully determined. In the quantum case the laws of quantum theory are assumed to be completely known but the state of the system is at least a maximal information but not a complete one. Hence the properties of the system can be determined only partially. In spite of the striking analogy between the classical and the quantum case there is a decisive difference: In classical physics the lack of information is only subjective and one could complete the knowledge of the physical laws in such a way that the temporal development and the final state of the system are completely determined. The modalities "necessary" and "possible" can then be eliminated in the scientific language. However in quantum physics the maximal information Φ about a system S can by no means be enlarged to a complete one. Hence there are always some observables the measuring values of which are *objectively undetermined* and thus the modalities "necessary" and "possible" are indispensable in the language of quantum physics.

2.2. *The definition of quantum logical modalities*

The preceding explanation of the concepts "necessary" and "possible" in quantum mechanics serves merely as an intuitive illustration. We are now going to define these concepts within the framework of the formal language of quantum physics which has been developed in Section 1. For this purpose we consider a system $S = S(W)$ with the preparation W. In general W will be a proposition which contains material propositions and an arbitrary but finite number of logical connectives. Hence the general proof procedure of proposition W is given by the material dialog game D_m. If for

a system $S(W)$ the truth of a proposition A can be demonstrated with *certainty*, this proposition A will be called "necessary with respect to W", which will be denoted by $\Delta_W A$. Since the most general proof procedure of a proposition A is given by a material dialog, a proposition A is necessary with respect to W if and only if there is a *strategy of success* for A relative to the preparation W within the material dialog game. Hence we define

$$\Delta_W A \rightleftharpoons W \vdash_{D_m} A.$$

Instead we could also define $\Delta_W A$ by the equivalent postulate, that there is a *strategy of success* for the material implication $W \rightarrow A$ within the material dialog game, i.e.,

$$\Delta_W A \rightleftharpoons \vdash_{D_m} W \rightarrow A \rightsquigarrow W \vdash_{D_m} A.$$

It is obvious that this definition is in accordance with the above mentioned requirements for the concept of "necessity". The truth of a proposition A can be demonstrated with *certainty* if and only if for any system $S^{(i)}(W)$ with preparation W the proposition A can be successfully defended within a material dialog which means that there is a *strategy of success* for A relative to W. Furthermore a material dialog of a dialog-definite proposition A consists partly of the experimental proofs of material propositions contained in A and partly of formal dialogic strategies. Hence the success of a D_m-dialog is based on formal strategies and on contingent relations between material propositions. Whereas the formal strategies used in a D_m-dialog correspond to the laws of the formal quantum logic Q_{eff} and Q, the contingent relations between material propositions follow from the laws $\Gamma_i^{(q)}$ of quantum theory, which are incorporated into the material dialog game only implicitly.

The contingent laws $\Gamma_i^{(q)}$ of quantum theory which are relevant here are relations of the form $M_A \leqslant M_B$ or $P_A = P_A P_B$, respectively. For relations of this kind we know that there is a strategy of success for $A \rightarrow B$ in D_m if and only if $M_A \leqslant M_B$ holds, i.e.,

$$M_A \leqslant M_B \rightsquigarrow \vdash_{D_m} A \rightarrow B.$$

In order to incorporate these *factual* implications[3] into the formal dialog game, we extend our definition for the implication "$\leqslant$" due to

$$A \leqslant B \rightleftharpoons \vdash_{D_m} A \to B.$$

A material dialog about $W \to A$ can then be replaced by a formal dialog with all *factual laws* $A_j \leqslant B_j$ as hypothesis

$$\vdash_{D_m} W \to A \frown \bigwedge_j (A_j \to B_j) \vdash_{D_f} W \to A.$$

This means that a materially true implication $W \leqslant A$ can be derived within the calculus Q_{eff} or Q respectively if the *factual implications* $A_j \leqslant B_j$ are added to the calculus as additional beginnings; i.e., we have

$$A_1 \leqslant B_1', A_2 \leqslant B_2', \ldots, A_n \leqslant B_n \vdash_{Q_{eff}} W \leqslant A.$$

If once the definition of the modality of necessity is established in this way as a material meta-proposition one can easily obtain further modalities by means of $\Delta_W A$ and the meta-logical connectives $\curlywedge$, $\curlyvee$, $\Rightarrow$ and $\exists$. Here we define three further modalities, the "possibility" $\nabla_W A$ and the "contingency" $\between_W A$, which are known from the traditional modal logic, and the "objectivity" $\square_W A$ which is of special importance for quantum mechanical propositions.

A is possible with respect to W: $\quad \nabla_W A \rightleftharpoons \exists \Delta_W \neg A,$

A is contingent with respect to W: $\quad \between_W A \rightleftharpoons \exists \Delta_W A \curlywedge \exists \Delta_W \neg A,$

A is objective with respect to W: $\quad \square_W A \rightleftharpoons \Delta_W A \curlyvee \Delta_W \neg A.$

If classical logic is applied to meta-propositions it follows that

$$\exists \square_W A \equiv \exists \{\Delta_W A \curlyvee \Delta_W \neg A\} \equiv$$

$$\exists \Delta_W A \curlywedge \exists \Delta_W \neg A \equiv \between_W A,$$

i.e., the contingency of a proposition A is equivalent to its "non-objectivity" with respect to W.

2.3. *The commensurability relation*

The modality of objectivity $\square_W A$ is intimately related to the commensurability relation $W \sim A$ between the propositions W and A, which can be expressed within the calculus Q by

$$W \sim A \frown W \leqslant (W \wedge A) \vee (W \wedge \neg A).$$

The proposition W is said to be *maximal* if $W \neq \Lambda$ and if it fulfills the relation

$$\Lambda \leqslant X \leqslant W \frown X = \Lambda \veebar X = W$$

which – in the framework of lattice theory – defines an *atom*. Using this definition the connection between objectivity and commensurability can then be expressed by the following theorems.

THEOREM 2.1. If the proposition W is *maximal*, then the commensurability of W and an arbitrary proposition A implies the objectivity of A with respect to W and vice versa, i.e.,

$$W \sim A \frown \Box_W A.$$

Proof. (a) $\frown$: Since proposition W is maximal we have

$$\Lambda \leqslant A \wedge W \leqslant W \frown A \wedge W = \Lambda \veebar A \wedge W = W.$$

From $A \wedge W = \Lambda$ it follows by means of $W \sim A$, i.e.,
$W \leqslant (A \wedge W) \vee (\neg A \wedge W)$ that $\neg A \wedge W = W$. Hence we obtain

$$W \leqslant A \veebar W \leqslant \neg A$$

which is by definition equivalent to $\Box_W A$.

(b) $\smile$: Let $\Box_W A \Rightarrow W \leqslant A \veebar W \leqslant \neg A$. Furthermore $W \leqslant A$ implies $W \leqslant (W \wedge A) \vee (W \wedge \neg A)$ and $W \leqslant \neg A$ implies $W \leqslant (W \wedge A) \vee (W \wedge \neg A)$, Hence we obtain

$$\Box_W A \frown W \leqslant (W \wedge A) \vee (W \wedge \neg A) \smile W \sim A.$$

In part (b) of this proof the maximality of proposition W has not been used. Therefore the second part of the theorem can be extended to arbitrary proposition W:

THEOREM 2.2. For arbitrary propositions W and A the objectivity of A with respect to W implies the commensurability of W and A, i.e.,

$$\Box_W A \frown W \sim A.$$

These theorems show that the concepts of "objectivity" and "non-objectivity" defined here are in fact in accordance with the use of these

terms in the framework of quantum theory. In particular it follows for the important concept of "non-objectivity" that a proposition A which is non-objective with respect to a maximal proposition W is incommensurable with it and vice versa. If however W is an arbitrary proposition it may happen (according to Theorem 2.2) that a proposition A which is not objective with respect to W, i.e., for which

$$\exists \Delta_W A \nprec \exists \Delta_W \neg A$$

holds, is nevertheless commensurable with W. The reason for this – at first glance – surprising result can be explained by the following example. Assume that the proposition W is given by

$$W = W_1 \vee W_2 \quad \text{with}$$

$$(2.1) \qquad W_1 \leqslant \neg W_2, \qquad W_1 \neq \Lambda \quad \text{and} \quad W_2 \neq \Lambda.$$

It is obvious that W is not maximal. In fact if W were maximal, we would have

$$\Lambda < W_1 \leqslant W_1 \vee W_2 \frown W_1 = W_1 \vee W_2$$

and thus $W_2 \leqslant W_1$. Since by assumption we have $W_2 \leqslant \neg W_1$, we would obtain $W_2 = \Lambda$ in contradiction with the premise. An analogous argument leads to $W_1 = \Lambda$.

Let us further assume that

$$(2.2) \qquad W_1 \leqslant A, \quad W_2 \leqslant \neg A$$

from which we obtain

$$(2.3) \qquad W_1 \sim A, \quad W_2 \sim A \quad \text{and} \quad W = (W_1 \vee W_2) \sim A,$$

i.e., propositions W and A are commensurable. It is easy to see that in addition to (2.3)

$$(2.4) \qquad \exists \Delta_W A \nprec \exists \Delta_W \neg A$$

hold, i.e., A is non-objective with respect to W. In order to prove this modality let us assume that $\Delta_W A$ would hold. From $\Delta_W A$ it follows that $W_2 \leqslant W_1 \vee W_2 \leqslant A$, from (2.2) one obtains $W_2 \leqslant \neg A$ and thus $W_2 = \Lambda$, in contradiction to (2.1). On the other hand, from $\Delta_W \neg A$ it follows that $W_1 \leqslant W_1 \vee W_2 \leqslant \neg A$, from (2.2) one obtains $W_1 \leqslant A$ and thus $W_1 = \Lambda$,

which is again in contradiction to (2.1). Hence the modality (2.4) holds in the present case.

These considerations hsow that for the special example which is characterized by (2.1) and (2.2) the commensurability $W \sim A$ of W and A is indeed compatible with the non-objectivity of A with respect to W. In the framework of quantum mechanics this example can easily be illustrated by a geometrical construction in Hilbert space. This will be done in the next subsection.

2.4. *Illustration of the modalities in Hilbert space*

The meaning of the quantum logical modalities can be further illustrated if we express the logical definitions in terms of quantum theory. In this theory the propositions W and A correspond to projection operators P_W and P_A respectively in the Hilbert space $\mathscr{H}(S)$ of the system in question. In particular, if $W = \phi$ is a maximal proposition the corresponding operator P_ϕ projects onto a one-dimensional subspace of $\mathscr{H}(S)$.

(a) Let $S = S(W)$ be a system with preparation W. Then we have according to the definition of $\Delta_W A$:

$$\Delta_W A \curvearrowleft P_W \leqslant P_A \curvearrowleft P_W = P_W P_A .$$

Furthermore from $P_W = P_W P_A$ it follows that $[P_W, P_A]_- = 0$, which means that W and A are commensurable, i.e.,

$$\Delta_W A \curvearrowleft [P_W, P_A]_- = 0.$$

However, the inverse of this relation does not hold in general.

(b) From the definition $\nabla_W A \rightleftharpoons \exists \Delta_W \neg A$ of the "possibility"-modality it follows that

$$\nabla_W A \curvearrowleft \exists \{P_W = P_W(1 - P_A)\} \curvearrowleft \exists \{P_W P_A = 0\} \curvearrowleft P_W P_A \neq 0.$$

From a physical point of view this modality does not seem to be very relevant since the inequality $P_W P_A \neq 0$ does not describe a physically interesting situation.

(c) From the definition

$$\Sigma_W A \rightleftharpoons \exists \Delta_W A \curlywedge \exists \Delta_W \neg A$$

of the "contingency"-modality, we obtain

$$\begin{aligned} \mathsf{X}_W A &\rightsquigarrow \neg(P_W = P_W P_A) \curlywedge \neg(P_W = P_W(1-P_A)) \\ &\rightsquigarrow P_W \neq P_W P_A \curlywedge P_W P_A \neq 0 \\ &\rightsquigarrow P_W \neq P_W P_A \neq 0. \end{aligned}$$

The meaning of this "contingency-modality" becomes more obvious if we recall the fact that the "contingency" $\mathsf{X}_W A$ is equivalent to the "non-objectivity" of A with respect to W, which is in fact very important in quantum theory.

(d) From the definition

$$\Box_W A \rightleftharpoons \Delta_W A \curlyvee \Delta_W \neg A$$

of the "objectivity"-modality one obtains

$$\begin{aligned} \Box_W A &\rightsquigarrow P_W = P_W P_A \curlyvee P_W = P_W P_{\neg A} \\ &\rightsquigarrow P_W = P_W P_A \curlyvee P_W P_A = 0. \end{aligned}$$

Since the equations $P_W = P_W P_A$ and $P_W P_A = 0$ both imply $[P_W, P_A]_- = 0$, we get

$$\Box_W A \rightsquigarrow [P_W, P_A]_- = 0$$

which means that the objectivity of A with respect to W implies the commensurability of W and P. The inverse of this relation cannot generally be shown except when W is a maximal proposition ϕ which corresponds to a one-dimensional subspace. In fact since P_ϕ is an atom, i.e., $0 \leqslant X \leqslant P_\phi$ implies $X = 0 \curlyvee X = P_\phi$, we obtain

$$[P_\phi, P_A]_- = 0 \rightsquigarrow P_\phi = P_\phi P_A \curlyvee P_\phi P_A = 0.$$

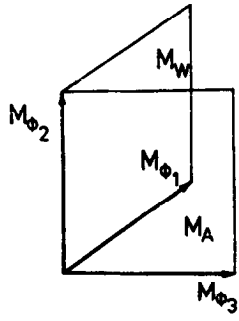

Fig. 1.

In order to show that this implication does not hold for an arbitrary proposition W we reformulate the above mentioned counter-example in terms of quantum theory. Let M_W be a two-dimensional subspace spanned by the orthogonal vectors M_{ϕ_1} and M_{ϕ_2}, and M_{ϕ_3} a third vector which is orthogonal to M_{ϕ_1} and M_{ϕ_2} (Fig. 1). Clearly the projection operators of M_W and of the 2-dimensional subspace M_A spanned by M_{ϕ_2} and M_{ϕ_3} commute, i.e., $[P_W, P_A]_- = 0$. However, neither the equation $P_W = P_W P_A$ nor $P_W P_A = 0$ holds in the present case since $P_W P_A = P_{\phi_1}$. Hence proposition A is not objective with respect to W.

3. THE MODAL LOGIC OF QUANTUM LOGIC

3.1. *Formally true modalities*

Since the modalities defined here can be reduced to $\Delta_W A$ we will consider only Δ-modalities in the following investigations. Starting from the "*elementary propositions*" $\Delta_W A$ one can then obtain "*compound modalities*" by means of the metalogical connectives. The proof of these compound modalities has to be carried out within the meta-logical dialog game $\bar{D}$, whereas the elementary modalities have to be proved outside of $\bar{D}$, i.e., within the object dialog game D_m. Since the proofs of elementary meta-propositions are still contained in the meta-dialog game we are dealing here – more precisely – with a "material meta dialog game" $\bar{D}_m$. As an example we consider the compound modality $\Delta_W A \Rightarrow \Delta_W(A \wedge B)$, the proof of which consists of a meta-dialog about the material meta-implication "$\Rightarrow$" and the object-dialogic proofs of the elementary modalities $\Delta_W A$ and $\Delta_W(A \wedge B)$. It is obvious that the truth of this compound modality depends on the proposition W.

There are, however, some compound modalities which can successfully be defended within a meta-dialog, irrespective of the proposition W to which elementary modalities are related. As an example we consider the meta-dialog for $\Delta_W A \Rightarrow \Delta_W B$ and object propositions A and B for which the *factual*[3] (empirical) implication $A \leqslant B \Leftrightarrow \vdash\!\!\!

we have made use of the *factual implication* $\vdash\!\!\!\overline{}_{D_m} A \to B$, which could have been presupposed as an hypothesis in the meta-dialog about $\Delta_W A \Rightarrow \Delta_W B$. – As a second example we consider the meta-dialog for $\Delta_W A \curlywedge \Delta_W(A \to B) \Rightarrow \Delta_W B$. After a successful proof of $\Delta_W A$ and $\Delta_W(A \to B)$ by the opponent, the proponent is challenged to prove $\Delta_W B$. This is, however, always possible, irrespective of the proposition W. In fact $\vdash\!\!\!\overline{}_{D_m} W \to A$ and $\vdash\!\!\!\overline{}_{D_m} W \to (A \to B)$ imply $\vdash\!\!\!\overline{}_{D_m} W \to (A \wedge (A \to B))$ and hence $\vdash\!\!\!\overline{}_{D_m} W \to B$. In this deviation we have made use of the *modus-ponens-law* for object propositions $(A \wedge (A \to B)) \leqslant B$ for which there exists even a *formal strategy* of success and which can thus be defended successfully in any material dialog.

In the *modal logic of quantum logic* such compound modalities are investigated which can be successfully defended within the meta-dialog game $\bar{D}$, irrespective of the object proposition W contained in it. The two examples just mentioned illustrate the method which can generally be used in order to prove "formally true modalities" of this kind: From the premises $\Delta_W A_1, \Delta_W A_2, \ldots$ one can obtain the respective conclusion $\Delta_W B$ if and only if the corresponding object-premises $A_1, A_2, \ldots$ (without Δ) imply the respective object-conclusion B either *factually* or even *formally*. Since in both cases there exists a strategy of success within the material dialog game these considerations can be comprehended in the following result.

THEOREM 3.1.[4] If in a position of a meta-dialog the opponent O has proved the modalities $\Delta_W A_1, \Delta_W A_2, \ldots, \Delta_W A_n$ and the proponent is challenged to prove $\Delta_W B$ – then P can win the meta-dialog for all W if and only if in the material object-dialog game D_m the proponent has a strategy for a dialog about B in which $A_1, A_2 \ldots A_n$ are available as propositions of O.

Proof. (a) Assume $\Delta_W A_1, \Delta_W A_2, \ldots, \Delta_W A_n$ are meta-propositions of O and P can win the meta-dialog about $\Delta_W B$ for any W. If we put $W = A_1 \wedge A_2 \ldots \wedge A_n$, the premises $\vdash\!\!\!\overline{}_{D_m} (A_1 \wedge \ldots \wedge A_n) \to A_j$ can even be shown to be true within the formal dialog game D_f. Hence P can prove $\vdash\!\!\!\overline{}_{D_m} (A_1 \wedge A_2 \ldots \wedge A_n) \to B$ without further premises, which means that P has a strategy of success for B in D_m if $A_1, A_2, \ldots, A_n$ are available as propositions of O.

(b) Conversely assume that P has a strategy of success for B in D_m if $A_1, A_2, \ldots, A_n$ are available as propositions of O. Then P can prove $\vdash_{D_m} (A_1 \wedge A_2 \ldots \wedge A_n) \rightarrow B$. Furthermore if $\vdash_{D_m} W \rightarrow A_j$ with $(j = 1, 2, \ldots, n)$ and $W \neq \Lambda$ are premises of O, then P can prove $\vdash_{D_m} W \rightarrow (A_1 \wedge A_2 \wedge \ldots \wedge A_n)$. Finally from $\vdash_{D_m} W \rightarrow (A_1 \wedge \ldots \wedge A_n)$ and $\vdash_{D_m} (A_1 \wedge \ldots \wedge A_n) \rightarrow B$ one obtains $\vdash_{D_m} W \rightarrow B$, completing the proof.

Due to this theorem a general proof procedure for formally true modalities is given, which reduces the proofs of the meta-proposition to the proofs of corresponding object propositions. Since the object propositions must be defended within the material quantum dialog game D_m, the provable object propositions are just those propositions which can be successfully defended within the formal dialog game D_f if the totality of all empirical (factual) laws $A_j \leqslant B_j$ has been presupposed as hypothesis. Due to the completeness theorems [2, 3] the same class of propositions can be deduced within the calculi Q_{eff} and Q if the empirical laws $A_j \leqslant B_j$ are added to these calculi as "factual beginnings". (The extensions of the calculi Q_{eff} and Q by the factual beginnings $A_j \leqslant B_j$ will be denoted here by $Q_{\text{eff}}^{(\mathcal{V})}$ and $Q^{(\mathcal{V})}$, respectively.) Consequently the structural properties of the quantum logical calculi $Q_{\text{eff}}^{(\mathcal{V})}$ and $Q^{(\mathcal{V})}$ – in particular the well known restrictions due to the possible incommensurability of object propositions – will be reproduced in the modal logic of quantum logic.

This modal logic of quantum logic can be represented most conveniently by a calculus $M(Q_{\text{eff}}^{(\mathcal{V})})$ or $M(Q^{(\mathcal{V})})$, respectively, which allows for the derivation of all formally true modalities. This calculus, which is rather complicated, will not be presented here (cf. [16]). It is obvious from the considerations mentioned above, that the calculi $M(Q_{\text{eff}}^{(\mathcal{V})})$ and $M(Q^{(\mathcal{V})})$ are essentially different from the corresponding modal calculi of intuitionistic and classical logic which are known from the literature.[5]

3.2. *Examples*

In order to illustrate the difference between the modal logic of quantum logic which can be summarized in the calculi $M(Q_{\text{eff}}^{(\mathcal{V})})$ and $M(Q^{(\mathcal{V})})$ and the corresponding modal calculi of ordinary logic, we shall consider some examples of compound modalities which are not valid in $M(Q_{\text{eff}}^{(\mathcal{V})})$ but which

are usually assumed to be true in ordinary modal logic. In order to demonstrate that a certain modality is not formally true we first investigate the respective meta-dialog applying Theorem (3.1). In a second step we illustrate the invalidity of the considered modality in special cases by a geometrical counter-example in Hilbert space.

(a) The modality $\Delta_W(A \vee B) \pi \Delta_W(\neg B) \Rightarrow \Delta_W A$ is not formally true. In fact the material meta-dialog about this modality reads

	O		P	
(0)	[ ]		$\Delta_W(A \vee B) \pi \Delta_W(\neg B) \Rightarrow \Delta_W A$	
(1)	$\Delta_W(A \vee B) \pi \Delta_W(\neg B)$	(0)	[ ]	
(2)	$\Delta_W(A \vee B)$!	⟨1⟩	1?	(1)
(3)	$\Delta_W \neg B$!	⟨1⟩	2?	(1)
(4)	[ ]		$\Delta_W B$	⟨0⟩
(5)	$\Delta_W B$?		[ ]	

According to Theorem (3.1) P can defend in $P(5)$ successfully, irrespective of W if and only if P has a strategy of success for $((A \vee B) \wedge \neg B) \rightarrow A$ in D_m. This is, however, generally not the case except when suitable factual relations between A and B hold, e.g., $A \leqslant \neg B$.

As a geometrical counter-example we consider a two dimensional Hilbert space spanned by the orthogonal one-dimensional subspaces M_B and $M_{\neg B}$. Furthermore, let M_A be an arbitrary one-dimensional subspace, and $M_W = M_\phi = M_{\neg B}$. Clearly we have $M_\phi \in M_A \cup M_B$ and $M_\phi \in M_{\neg B}$ (Fig. 2). However, since $M_\phi \notin M_A$ the general validity of the modality in question is disproved.

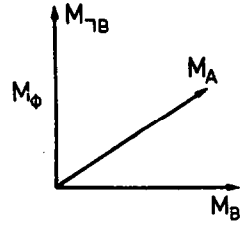

Fig. 2.

(b) The modality $\Delta_W A \Rightarrow \Delta_W(B \to A)$ is not formally true. The material meta-dialog about this modality reads:

	O		P	
(0)	[ ]		$\Delta_W A \Rightarrow \Delta_W(B \to A)$	
(1)	$\Delta_W A$	(0)	[ ]	
(2)	$\Delta_W A$!	⟨1⟩	$\Delta_W A$?	(1)
(3)	[ ]		$\Delta_W(B \to A)$	⟨1⟩
(4)	$\Delta_W(B \to A)$?	(3)	[ ]	

Here the proponent cannot defend in $P(5)$. According to Theorem (3.1) he could continue if there would be a strategy of success for the object-proposition $A \to (B \to A)$ in D_m. This is, however, the case if and only if the propositions A and B are formally or at least factually commensurable.

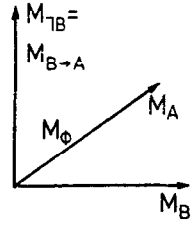

Fig. 3.

As a geometrical counter-example we consider again the two-dimensional Hilbert space spanned by the orthogonal one-dimensional subspaces M_B and $M_{\neg B}$. Let M_A be an arbitrary one-dimensional subspace and $M_W = M_\phi = M_A$ (Fig. 3). Since in the calculus Q we have the formal equivalence $B \to A = \neg B \vee (B \wedge A)$ we find

$$M_{B \to A} = M_{\neg B} \cup (M_B \cap M_A) = M_{\neg B}.$$

Clearly we have $M_\phi \in M_A$ but since in this special case $M_\phi \notin M_{B \to A}$ the general validity of the modality $\Delta_W A \Rightarrow \Delta_W(B \to A)$ is disproved.

(c) The modality $\Delta_W(A \to B) \pi \Delta_W(\neg B) \Rightarrow \Delta_W(\neg A)$ is not formally true, even if we use Q instead of Q_{eff}. In this case we can rewrite the modality in the form

$$\Delta_W(\neg A \vee (A \wedge B)) \pi \Delta_W(\neg B) \Rightarrow \Delta_W(\neg A).$$

According to Theorem (3.1) this modality can be successfully defended within the material meta-dialog game $\bar{D}_m$ if and only if the implication

$$(\neg A \vee (A \wedge B)) \wedge \neg B \leqslant \neg A$$

can be derived in the framework of the extended calculus $Q^{(\sigma)}$ of full quantum logic. This is however not the case, except when a suitable factual relation between A and B holds, e.g., $\neg A \leqslant \neg B$. A geometrical counter-example for this modality cannot be constructed in a space of three or less dimensions.

These three counter-examples show that the modal logic of quantum logic is actually quite different from the modal logic of ordinary (intuitionistic or classical) logic, which is known from the modal calculi M, S4, etc. [6]. In fact, the implications $(A \vee B) \wedge \neg B \leqslant A, A \leqslant B \to A$ and $(A \to B) \wedge \neg B \leqslant \neg A$ which are needed for the proofs of the modalities are formally true in ordinary logic. Thus we arrive at the interesting result that the *formal meta-logic* of quantum logic is equivalent to the meta-logic of ordinary logic, but that the *modal logic* of quantum logic is essentially different from the corresponding classical structure.

3.3. *The principle of excluded middle*

The modality $\Delta_W A \curlyvee \Delta_W B \Rightarrow \Delta_W (A \vee B)$ is formally true – within the framework of the modal logic of ordinary logic as well as in the modal logic of quantum logic. In fact for the propositions $A \to (A \vee B)$ and $B \to (A \vee B)$ there is a strategy of success even within the formal quantum dialog game D_f. However the inverse of this modality, namely

$$(3.1) \qquad \Delta_W (A \vee B) \Rightarrow \Delta_W A \curlyvee \Delta_W B$$

is not generally valid, even not in the modal logic $M(L_C)$ of classical logic L_C.[6] This can be seen as follows. If we put $W = A \vee B$, the premise of the material meta-implication (3.1) is formally true. However, none of the possible conclusions, i.e., $A \vee B \leqslant A$ or $A \vee B \leqslant B$ is also formally true. In fact these relations are false if none of the implications $A \leqslant B$ or $B \leqslant A$ holds.

There is, however, an important class of propositions W for which the modality (3.1) is formally true at least within the modal logic $M(L_C)$ of classical logic. This is the case if W is a maximal proposition, i.e., if

$$W \neq \Lambda, \quad \Lambda < X < W \frown X = \Lambda \veebar X = W$$

holds. From this condition one obtains for an arbitrary proposition C

$$\Lambda < W \wedge C < W \frown W \wedge C = \Lambda \veebar W \wedge C = W$$

and thus

$$(3.2) \qquad W < \neg C \veebar W < C,$$

i.e., every C is "*objective*" with respect to a maximal proposition W. Using this objectivity property one finds that within the classical logic L_C the implication $W < A \vee B$ implies $W < A \veebar W < B$. In fact, $W < A \vee B$ implies $\exists\{W < \neg(A \vee B)\}$ and thus $\exists\{W < \neg A\} \veebar \exists\{W < \neg B\}$ from which one obtains by means of (3.2) $W < A \veebar W < B$. Hence for the class of atomic propositions W the modality (3.1) is generally true.

In the modal logic of *quantum logic* the modality (3.1) is of course not formally true either. However, in the quantum case the modality (3.1) is not even valid for the class of maximal propositions W. Namely, since within the calculi Q_{eff} and Q, $W \wedge C = \Lambda \veebar W \wedge C = W$ does not imply $W < \neg C \veebar W < C$, the above mentioned proof is not conclusive in quantum logic. Furthermore a counter-example can easily be constructed which demonstrates the invalidity of (3.1) for special propositions. Let $W \neq \Lambda$ be maximal and the propositions A, B chosen such that the relations

$$(3.3) \qquad \Lambda = A \wedge W = B \wedge W < W < A \vee B$$

hold.[7] Hence for this choice the premise $\Delta_W(A \vee B)$ of (3.1) is fulfilled. However none of the two possible conclusions $\Delta_W A$ and $\Delta_W B$ holds in the present case. In fact $W < A$ or $W < B$ would imply $W < W \wedge A = \Lambda$ or $W < W \wedge B = \Lambda$ respectively in contradiction to the assumption. It is obvious that this consequence can not be avoided due to the maximality of proposition W.

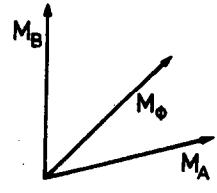

Fig. 4.

This special choice of propositions W, A and B can be illustrated geometrically in a two-dimensional Hilbert-space spanned by the one-dimensional subspaces M_A and M_B (Fig. 4). If $M_W = M_\phi$ is given also by a one-dimensional subspace – which is of course an atom – we have $M_\phi \in M_A \cup M_B$, but generally none of the relations $M_\phi \in M_A$ and $M_\phi \in M_B$ holds, except when W is commensurable with A or with B.

A special case of the modality (3.1) which is of particular interest is given by $B = \neg A$, i.e.,

$$\Delta_W(A \vee \neg A) \Rightarrow \Delta_W A \veebar \Delta_W \neg A. \tag{3.4}$$

This modality is again not formally true, neither in classical logic L_C nor in full quantum logic Q. Since in L_C and Q the "*principle of excluded middle*" $\mathrm{V} \leqslant A \vee \neg A$ generally holds, the premise $\Delta_W(A \vee \neg A)$ of (3.4) is fulfilled for arbitrary W. However, the conclusion $\Delta_W A \veebar \Delta_W \neg A$ is not formally true, neither for L_C nor for Q. This means that a given proposition A is not "objective" with respect to every proposition W.

As mentioned above, in classical logic L_C the modality (3.1) is formally true for the restricted class of maximal propositions W. Consequently also (3.4) holds in L_C for maximal W, which means that in the modal logic of classical logic a given proposition A is "objective" with respect to every maximal proposition W. This result has the following important consequenc consequence: Since a free Boolean propositional lattice $L_B = L_B(\mathscr{G}_e)$, which is generated by a finite set $\mathscr{G}_e$ of value definite elementary propositions, is always atomic, it follows that for *every* proposition $W \in L_B$ there is an atom $W_0 \leqslant W$ such that a given proposition A is objective with respect to W_0, i.e., that

$$\Box_{W_0} A \frown \Delta_{W_0} A \veebar \Delta_{W_0} \neg A; \quad W_0 \leqslant W$$

holds.[8]

However, this result is not valid in the modal logic $M(Q_{eff})$ or $M(Q)$ of quantum logic. If W is a maximal proposition it follows again that $A \wedge W = \bigwedge \veebar A \wedge W = W$ holds, but in quantum logic (Q_{eff} or Q) this result does not imply the validity of the "objectivity"-modality $\Delta_W A \veebar \Delta_W \neg A$. Therefore, the law $\Delta_W A \veebar \Delta_W \neg A$, sometimes and misleadingly called "principle of excluded middle",[9] does not hold in $M(Q)$ even for the class of maximal proposition W – in spite of the deducibility of $\mathrm{V} \leqslant A \vee \neg A$ in Q. – Since in terms of quantum theory the objectivity

$\Box_W A$ with respect to a maximal W is equivalent to the commensurability $[P_W, P_A]_- = 0$ of W and A, this means that a proposition A is not commensurable with any arbitrary maximal proposition (state vector) $W = \phi$.

The fact that the "objectivity"-modality,

$$\Delta_W A \curlyvee \Delta_W \neg A$$

is not generally valid in $M(Q)$ for maximal propositions W is indeed characteristic for the situation in quantum modal logic. Whereas the propositions A about a classical system are objective with respect to every atomic proposition W_0, the propositions A about a quantum system $S(\phi)$ are objective only with respect to those maximal propositions ϕ, which are commensurable with A. The lack of the general validity of $\Delta_W A \curlyvee \Delta_W \neg A$ for maximal propositions W in quantum modal logic and the fact that this modality has occasionally and misleadingly been interpreted as "*principle of excluded middle*" explains why some authors have claimed (e.g., Reichenbach [17], Destouches-Férrier [18]) that quantum logic is characterized by the invalidity of the principle of excluded middle. – It is obvious that this mode of expression is not in accordance with the present approach.

University of Cologne, Cologne, Germany

NOTES

[1] The material meta-implication "$\rightarrow$" has been denoted here by the same sign which is used for the denotation of rules in the calculus Q_{eff}. The justification for this terminology can be found in [1, p. 142] and in more detail in [4, p. 352].

[2] For the interpretation of the modalities "necessary" and "possible" by an incomplete determinism, some authors refer to Aristotle (De interpretatione IX), e.g., Anscombe [10] and Hintikka [11]. However, this interpretation has not been left unchallenged. By a careful analysis of the Greek text, D. Frede [12] arrives at the result that the modal interpretation of De interpretatione IX is untenable. Instead, the traditional interpretation which goes back to Boetius is found to be more adapted to Aristotle's text.

[3] A proposition, relation, etc. is said to be "*factually*" true if it is true for contingent reasons and can be verified empirically, but if it is not *formally* true. This terminology is in accordance with Carnap [14, p. 146] and [15, p. 12].

[4] This theorem can be traced back to Aristotle who mentions (*Analytica priora*, book I, Chapt. 8) in a rather vague form the relation between necessary and factual propositions.

[5] Actually one can construct only the calculi $M(Q_{eff})$ and $M(Q)$ which take account of all formally true implications. However, one could always extend these calculi by incorporating the factual implications of the field of objects considered.

[6] By L_C we denote here the calculus of classical propositional logic, which from an algebraic point of view corresponds to a Boolean lattice L_B. The relations between L_C and the calculi Q_{eff} and Q, respectively, can be found in [1, Chapt. II and V].

[7] Within the calculi Q_{eff} and Q these relations are not contradictory. In classical logic they would lead to a contradiction, since in L_C $A \wedge W = \bigwedge$ and $B \wedge W = \bigwedge$ imply $W \leqslant \neg A$ and $W \leqslant \neg B$, respectively, and thus it would follow that $W = \bigwedge$.

[8] In the framework of the modal interpretation of Aristotle's "*De interpretatione*" IX [10, 11] the invalidity of the formula $\Delta_W(A \vee \neg A) \rightarrow \Delta_W A \vee \Delta_W \neg A$ can be illustrated by Aristotle's famous example for proposition A: "*There will be a sea-battle tomorrow.*" Whereas $\Delta_W(A \vee \neg A)$ is true for every information W in the framework of the above mentioned (cf. note 2) incomplete determinism there is no information W available such that A or $\neg A$ is necessary with respect to W. By the assumption of an incomplete determinism one obviously excludes the case that W is a maximal proposition.

[9] It is obvious that this terminology is misleading since the "principle of excluded middle" in the object language is given by $\bigvee \leqslant A \vee \neg A$, whereas the "*tertium non datur*" of the meta-language reads $\Delta_W A \vee \exists \Delta_W A$.

BIBLIOGRAPHY

[1] Mittelstaedt, P., *Quantum Logic*, D. Reidel Publ. Co., Dordrecht, Holland, 1978.

[2] Stachow, E. W., 'The Completeness of Quantum Logic', *Journal of Philosophical Logic*, 5 (1976), 237–279.

[3] Mittelstaedt, P. and Stachow, E. W., 'The Principle of Excluded Middle in Quantum Logic', *Journal of Philosophical Logic* 7 (1978), 181–208.

[4] Stachow, E. W., 'Quantum Logical Calculi and Lattice Structures', *Journal of Philosophical Logic* 7 (1978), 347–386.

[5] Lorenzen, P., *Formal Logic*, D. Reidel Publ. Co., Dordrecht, Holland (1965).

[6] Hughes, G. E. and Cresswell, M. J., *An Introduction to Modal Logic*, London: Methuen and Co., Ltd. (1968).

[7] Hintikka, J., *Models for Modalities*, D. Reidel Publ. Co., Dordrecht, Holland (1969).

[8] Schütte, K., *Vollständige Systeme modaler und intuitionistischer Logik*, Springer-Verlag, Berlin, Heidelberg, New York (1968).

[9] Lorenzen, P. und Schwemmer, O., *Konstruktive Logik, Ethik und Wissenschaftstheorie*, Bibliographisches Institut, Mannheim (1972).

[10] Anscombe, G. E. M., 'Aristotle and the Sea Battle', *Mind* (1956), pp. 1–10.

[11] Hintikka, J., 'The Once and Future Sea Fight. Aristotle's Discussion of Future Contingents in De Interpretatione IX', *The Philosophical Review* (1964), pp. 461–492.

[12] Frede, Dorothea, *Aristoteles und die Seeschlacht*, Vandenhoeck und Ruprecht, Göttingen (1970).

[13] Mittelstaedt, P., *Philosophical Problems of Modern Physics*, D. Reidel Publ. Co., Dordrecht, Holland (1976).

[14] Carnap, R., Induktive Logik und Wahrscheinlichkeit, (W. Stegmüller, ed.) Springer-Verlag, Wien (1959).

[15] Carnap, R., *Meaning and Necessity*, University of Chicago Press, Chicago (1956).

[16] Burghardt, F. J., (forthcoming).
[17] Reichenbach, H., *Philosophic Foundations of Quantum Mechanics*, University of California Press, Berkeley and Los Angeles (1944, 1965).
[18] Destouches-Férrier, P., *La Structure de Théories Physique*, Press Universitaires de France, Paris (1951).
[19] Dishkant, H., 'Semantics of the Minimal Logic of Quantum Mechanics', *Studia Logica XXX* (1972), pp. 23–26.
[20] Goldblatt, R. I., 'Semantic Analysis of Orthologie', *J. Phil. Logic* 3 (1974), pp. 19–35.
[21] Dalla Chiara, M. L., 'Quantum Logic and Physical Modalities', *J. Phil. Logic* 6 (1977), pp. 391–404.
[22] Kripke, S., 'Semantical Considerations in Modal Logic', *Acta Philosophica Fennica* 16 (1963), pp. 83–94.